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Problem Set # 9

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Exercise 1:
Find the Jordan canonical form for $L_A : \mathbb{C}^3 \to \mathbb{C}^3$ with

$$A = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -4 \\ 3 & -1 & 0 \end{pmatrix}$$

Exercise 2: Let V be the subspace $\mathbb{R} - Span\{1, t, t^2, e^t, te^t\}$ in the infinite dimensional vector space $C^\infty(\mathbb{R})$ = all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Let $T = \frac{d}{dt}$.

1. Prove that this linear operator $T : C^\infty \to C^\infty$ leaves V invariant and can be viewed as a map $T : V \to V$.
2. Find the Jordan canonical form of $T : V \to V$ and find a basis for each generalized eigenspace.

Exercise 3: Find a basis $\mathcal{Y}$ that puts $L_A : \mathbb{C}^3 \to \mathbb{C}^3$ with

$$A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix}$$

into Jordan canonical form, and find the transition matrix Q such that the Jordan form

$$[L_A]_{\mathcal{Y}} = QAQ^{-1} = Q[L_A]_{\mathcal{X}}Q^{-1},$$

for $\mathcal{X} = \{e_1, e_2, e_3\}$ the standard basis in $\mathbb{C}^3$.

Exercise 4: Find a Jordan bases that put $T = \frac{d}{dt}$ into a Jordan canonical form on $V = \mathbb{R} - Span\{e^t, te^t, t^2e^t, e^{2t}\}$.

Exercise 5:
Let V be a vector space. Define its complexification as $V_{\mathbb{C}} = V + iV$, and we define on $V_{\mathbb{C}}$ two operations, for any $x = x_1 + ix_2$, $y = y_1 + iy_2 \in V_{\mathbb{C}}$ and $\lambda = a + ib \in \mathbb{C}$,

$$\begin{cases} x + y = (x_1 + y_1) + i(x_2 + y_2) & (addition) \\ \lambda x = (ax_1 - bx_2) + i(ax_2 + bx_1) & (scalar\ multiplication) \end{cases}$$

Prove that $V_{\mathbb{C}}$ is a vector space over $\mathbb{C}$.

Exercise 6:
Let V be a complex finite dimensional inner product space, $T : V \to V$ a linear operator, and $\mathcal{X}$ a basis (not necessarily orthonormal) such that $A = [T]_{\mathcal{X}}$ is upper triangular. (Such basis exist, as in the Jordan canonical form.) Let $\mathcal{N}$ be the ON basis obtained from $\mathcal{X}$ via the Gram-Schmidt process. Prove that $[T]_{\mathcal{N}}$ is again upper triangular.

Exercise 7:
Let $T : V \to V$ be diagonalizable over $\mathbb{F}$, with spectral decomposition

$$T = \sum_{\lambda \in Sp_{\mathbb{F}}(T)} \lambda P_\lambda$$

where P_λ is the projection onto $E_\lambda(T)$ along $\oplus E_\mu(T)$. Explain why, for each projection P_λ there is a polynomial $f \in \mathbb{F}[x]$, not necessarily unique such that P_λ is a linear combination of powers $f(T) = \sum_{j=0} c_j T^j$.

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Exercise 8:
We say that $T : V \to V$ a linear operator on a finite dimensional vector space V is

strictly positive definite (we write $T > 0$), if $T^* = T$ and $(T(v), v) > 0$, for all $v \neq 0$. (so all eigenvalues are real and $\lambda > 0$). These operators are always invertible.
If $T = T^*$ on a complex inner product space V, prove that

1. $e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k$ is self adjoint.
2. te^T is strictly positive definite.
3. the map $Exp : T \to e^T$ is one-to-one from the space of self adjoint operators $\mathcal{H} = \{T : T = T^*\}$ into the space $\mathcal{P}$ of strictly positive definite operators.
4. $Exp : \mathcal{H} \to \mathcal{P}$ is surjective (hence a bijection).

Thus every $B \in \mathcal{P}$ has a unique self-adjoint logarithm $A = log(B)$ such that $e^A = B$.
Hint: In 4., consider how you might recover the spectral decomposition of a self-adjoint operator T if you know the spectral decomposition of e^T